Revising the rules of differentiation

* Chain rule : If $y = f[g(x)]$ then $\frac{dy}{dx} = g'(x) f'(x)$

In simple words, if you have a function of a function then you need to apply the chain rule

e.g. $y = (2x+1)^5$

$\frac{dy}{dx} = 5(2x+1)^4 \cdot 2 = 10(2x+1)^4$

Get the power down, deduct one from the power and multiply by the derivative of the bracket

$y = \sin^5 x = (\sin x)^5$ $\qquad \frac{dy}{dx} = 5(\sin x)^4 \cos x = 5\cos x \sin^4 x$

You can see this analytically if you consider $u = \sin x$

Then $y = u^5$. Using the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5u^4 \cdot \cos x = 5\sin^4 x \cos x$$

$y = e^{x^2+5x+3}$ $\qquad \frac{dy}{dx} = (2x+5)\, e^{x^2+5x+3}$

You can see this analytically if you consider $u = x^2+5x+3$

Then $y = e^u$. Using the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot (2x+5) = (2x+5)e^{x^2+5x+3}$$

$y = \sin(x^2+5x)$ $\qquad \frac{dy}{dx} = (2x+5)\cos(x^2+5x)$

You can see this analytically if you consider $u = x^2+5x$

Then $y = \sin u$. Using the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \cdot (2x+5) = (2x+5)\cos(x^2+5x)$$

$y = \ln(x^2+\sin x)$ $\qquad \frac{dy}{dx} = \frac{2x+\cos x}{x^2+\sin x}$

You can see this analytically if you consider $u = x^2+\sin x$

Then $y = \ln u$. Using the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (2x+\cos x) = \frac{2x+\cos x}{x^2+\sin x}$$

* Product rule: If $y = f(x)\,g(x)$ then $\frac{dy}{dx} = f'(x)\,g(x) + f(x)\,g'(x)$

In simple words, if you have to differentiate a product of two functions (ie multiplication) then you differentiate the first, keep the second + keep the first, differentiate the second.

eg. $y = \underset{f(x)}{x^2}\,\underset{g(x)}{\sin x}$

$$\frac{dy}{dx} = \underset{f'(x)}{2x}\,\underset{g(x)}{\sin x} + \underset{f(x)}{x^2}\,\underset{g'(x)}{\cos x}$$

$y = xe^x \qquad \frac{dy}{dx} = e^x + xe^x$

$y = x^2\cos^2 x \qquad \frac{dy}{dx} = 2x\cos^2 x + x^2\, 2\cos x \sin x$

chain rule to find the derivative of $\cos^2 x$

$y = (2x+1)^4(3x+5)^2 \qquad \frac{dy}{dx} = 4(2x+1)^3 \cdot 2 \cdot (3x+5)^2 + (2x+1)^4 \cdot 2(3x+5) \cdot 3$

$$= 8(2x+1)^3(3x+5)^2 + 6(2x+1)^4(3x+5)$$

$y = e^{\sin x} \cdot \ln(x^2+5) \qquad \frac{dy}{dx} = \cos x\, e^{\sin x} \ln(x^2+5) + e^{\sin x}\,(2x)\,\ln(x^2+5)$

* Quotient rule: If $y = \frac{f(x)}{g(x)}$ then $\frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

In simple words, if you have to differentiate a quotient of two functions (ie division) then you differentiate the top, keep the bottom − keep the top, differentiate the bottom and all over the bottom squared.

eg. $y = \frac{x^2 \; f(x)}{\sin x \; g(x)}$

$$\frac{dy}{dx} = \frac{\overset{f'(x)}{2x}\,\overset{g(x)}{\sin x} - \overset{f(x)}{x^2}\,\overset{g'(x)}{\cos x}}{\underset{g^2(x)}{\sin^2 x}}$$

$y = \frac{(2x+1)^4}{\ln x} \qquad \frac{dy}{dx} = \frac{4(2x+1)^3 \cdot 2\ln x - (2x+1)^4 \frac{1}{x}}{(\ln x)^2}$

$y = \frac{e^{x^2+3x}}{\sin^2 x} \qquad \frac{dy}{dx} = \frac{(2x+3)\,e^{x^2+3x}\sin^2 x - e^{x^2+3x} \cdot 2\sin x\cos x}{(\sin^2 x)^2}$

$y = \frac{x+\cos x}{e^x+1} \qquad \frac{dy}{dx} = \frac{(1-\sin x)(e^x+1) - (x+\cos x)e^x}{(e^x+1)^2}$

$y = \ln\left(\frac{x+1}{\sin x}\right) \qquad \frac{dy}{dx} = \frac{\sin x - (x+1)\cos x}{\sin^2 x} \cdot \frac{1}{\frac{x+1}{\sin x}} = \frac{\sin x - (x+1)\cos x}{(x+1)\sin x}$ (HARD) Chain rule, with quotient rule

Can I skip the quotient rule and always get away with just knowing the product rule? Yes, BUT most of the times it will lead to something which is too messy and so increase our risk of making a mistake.