C2 - Chapter 7 - Geometric sequences and series - Summary

* A geometric sequence is a sequence of numbers where each subsequent term is generated by multiplying the previous one by the common ratio, r.
* A geometric sequence may be described fully in terms of the first term, a and the common ratio, r.
* $U_n = ar^{n-1} \qquad n \geq 1$
* $\dfrac{U_2}{U_1} = \dfrac{U_3}{U_2} = \dfrac{U_4}{U_3} = \ldots = r$
* $S_n = \dfrac{a(1-r^n)}{1-r}$

Please note that you need to know the derivation of the formula for S_n.

$$S_n = U_1 + U_2 + U_3 + U_4 + \ldots + U_n$$

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$$

$$- \quad rS_n = \quad ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$$

Multiply S_n by r

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$\therefore S_n = \frac{a(1-r^n)}{1-r}$$

* A convergent geometric series is a geometric series with $-1<r<1$ (or $|r|<1$).
* If a geometric series is convergent, then it has a sum to infinity which is given by $S_\infty = \dfrac{a}{1-r}$.

Please note that there is nothing stopping us from calculating $\dfrac{a}{1-r}$ for any geometric series, however it will be the sum to infinity only if $|r|<1$.

* Recall from C1 the Σ notation which indicates sum.

e.g. Find $\sum_{i=5}^{20} 20\times(1.2)^i$.

Start by writing out the first few terms (i.e. put $i=5$, then $i=6$, etc)

$$\Rightarrow \sum_{i=5}^{20} 20\times(1.2)^i = 20\times1.2^5 + 20\times1.2^6 + 20\times1.2^7 + \ldots + 20\times1.2^{20}$$

$$= 49.7664 + 59.71968 + \ldots$$

Hence, $a = 49.7664$ $r = 1.2$

$$\sum_{i=5}^{20} 20\times(1.2)^i = S_{16} \quad 16 = 20-5+1$$

$$= \frac{49.7664\left(1-1.2^{16}\right)}{1-1.2} = 4351.68 \text{ (to 2 dp's)}$$

* With verbal problems, invest time in understanding the situation and formulating your answer in terms of the question being asked.

e.g. At the beginning of 2013 I invested €2000 at a rate of 2.5%. At the beginning of every subsequent year I invest an extra €2000.

a) Find the total amount the investment is worth at the end of 2018

b) By the end of which year will the investment be worth at least €20000?

a) Investment at end of

	1st year	2nd year	3rd year ...	nth year
	2013	2014	2015 ...	
	2000×1.025	$2000\times1.025^2 + 2000\times1.025$	$2000\times1.05^3 + 2000\times1.025^2 + 2000\times1.025$	$2000\times1.05^n + \ldots + 2000\times1.05^3 + 2000\times1.025^2 + 2000\times1.025$
	$U_1 = S_1$	$U_2+U_1 = S_2$	$U_3+U_2+U_1 = S_3$	$U_n+\ldots+U_2+U_1 = S_n$

$\therefore$ The total amount the investment is worth at the end of the nth year is given by $S_n = \frac{a(1-r^n)}{1-r}$ where $a = 2000 \times 1.025 = 2050$ and $r = 1.025$

Now, 2018 is the 6^{th} year $\Rightarrow S_6 = \frac{2050(1-1.025^6)}{1-1.025} = €13094.86$

b) We need to solve $S_n > 20000$

$$\Rightarrow \frac{2050(1-1.025^n)}{1-1.025} > 20000$$

$$\frac{2050(1-1.025^n)}{-0.025} > 20000$$

$$-82000(1-1.025^n) > 20000$$

$$-82000 + 82000(1.025^n) > 20000$$

$$82000(1.025^n) > 102000$$

$$1.025^n > \frac{102000}{82000}$$

$$n \log 1.025 > \log(102/82)$$

$$n > \frac{\log(102/82)}{\log 1.025}$$

$$n > 8.84 \Rightarrow n = 9$$

Remember that if you divide or multiply by a negative number then you also need to reverse the inequality direction.

$\therefore$ By the end of the 9^{th} year, ie 2021.