C2 - Chapters 9 and 11 - Differentiation and Integration - Extra practice - Solutions

1. i) $y = 2x^3 + \sqrt{x} + \frac{x^2+2x}{x^2} = 2x^3 + x^{1/2} + 1 + 2x^{-1}$

$\frac{dy}{dx} = 6x^2 + \frac{1}{2}x^{-1/2} - 2x^{-2}$

ii) $\int_1^4 \left(\frac{x}{2} + \frac{1}{x^2}\right) dx = \int_1^4 \left(\frac{x}{2} + x^{-2}\right) dx = \left[\frac{x^2}{4} + \frac{x^{-1}}{-1}\right]_1^4 = \left[\frac{x^2}{4} - \frac{1}{x}\right]_1^4$

$= \left(\frac{4^2}{4} - \frac{1}{4}\right) - \left(\frac{1^2}{4} - \frac{1}{1}\right) = 9/2$

2. a) Volume = length · width · height

$= (50-2x)(40-2x)x = (2000 - 180x + 4x^2)x$

$= 4x(x^2 - 45x + 500)$ AS REQUIRED

b) $0 < x < 20$

c) $V = 4x^3 - 180x^2 + 2000x$

$\frac{dV}{dx} = 12x^2 - 360x + 2000$

Maximum $\Rightarrow \frac{dV}{dx} = 0 \Rightarrow 12x^2 - 360x + 2000 = 0$

$3x^2 - 90x + 500 = 0$

$x = \frac{90 \pm \sqrt{(-90)^2 - 4(3)(500)}}{2(3)} = 22.6, 7.36$

Reject (22.6)

d) $V_{max} = 4\cdot(7.36\ldots)^3 - 180(7.36\ldots)^2 + 2000(7.36\ldots) = 6564\ cm^3$

e) $\frac{d^2V}{dx^2} = 24x - 360$ $\quad \left.\frac{d^2V}{dx^2}\right|_{x=7.36\ldots} = -183 < 0 \quad \therefore$ Maximum

3. a) $y = x^3 - 6x^2 + 9x = x(x^2 - 6x + 9) = x(x-3)^2$ $\quad A(3,0)$

b) $\frac{dy}{dx} = 3x^2 - 12x + 9$ $\quad$ At B, $\frac{dy}{dx} = 0$ $\quad 3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x = 1$ OR $x = 3$

$y = 4$ $\quad y = 0$

$\therefore B(1,4)$

c) $R = \int_0^3 x^3 - 6x^2 + 9x\ dx = \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{9x^2}{2}\right]_0^3 = \left(\frac{3^4}{4} - \frac{6\cdot 3^3}{3} + 9\cdot\frac{3^2}{2}\right) - 0 = \frac{27}{4}$

4. a) $f(x) = x^3 - 6x^2 + 5x = x(x^2 - 6x + 5) = x(x-1)(x-5)$

b) At A $x = 1$ At B $x = 5$

c) $\frac{dy}{dx} = 3x^2 - 12x + 5$ $\left.\frac{dy}{dx}\right|_{x=1} = 3 \cdot 1^2 - 12 \cdot 1 + 5 = -4$

d) $R = \int_0^1 x^3 - 6x^2 + 5x \, dx = \left[\frac{x^4}{4} - 6\frac{x^3}{3} + 5\frac{x^2}{2}\right]_0^1 = 3/4$

$S = \int_1^5 x^3 - 6x^2 + 5x \, dx = \left[\frac{x^4}{4} - 6\frac{x^3}{3} + 5\frac{x^2}{2}\right]_1^5$

$= \left(\frac{5^4}{4} - \frac{6.5^3}{3} + \frac{5.5^2}{2}\right) - \left(\frac{1^4}{4} - \frac{6.1^3}{3} + \frac{5.1^2}{2}\right) = -32$

$\therefore$ Combined area $= 3/4 + 32 = 32\,3/4$

5 a) $\frac{dy}{dx} = 3x^2 - 14x + 15$

b) At Q, $\frac{dy}{dx} = 0$ $\Rightarrow$ $3x^2 - 14x + 15 = 0$

$(3x - 5)(x - 3) = 0$

$x = 5/3$ OR $x = 3$

$y = 12$

$\therefore Q(3, 12)$

c) At P, $x = 1$, $y = 1^3 - 7 \cdot 1^2 + 15 \cdot 1 + 3 = 12$

Since P and Q have the same y-coordinate, PQ is parallel to the x-axis.

d) Area $= \int_1^3 x^3 - 7x^2 + 15x + 3 \, dx - 2 \times 12$ ← area of rectangle

$= \left[\frac{x^4}{4} - \frac{7x^3}{3} + \frac{15x^2}{2} + 3x\right]_1^3 - 24$

$= \left(\frac{3^4}{4} - \frac{7 \cdot 3^3}{3} + \frac{15 \cdot 3^2}{2} + 3 \cdot 3\right) - \left(\frac{1^4}{4} - \frac{7 \cdot 1^3}{3} + \frac{15 \cdot 1^2}{2} + 3 \cdot 1\right) - 24 = 4/3$

6. a) $x + 1 = 6x - x^2 - 3$

$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$

$x = 1$ OR $x = 4$

$y = 2$ $y = 5$ $\therefore A(1, 2)$ $B(4, 5)$

b) $R = \int_1^4 6x - x^2 - 3 - (x+1)\,dx = \int_1^4 5x - x^2 - 4\,dx = \left[\frac{5x^2}{2} - \frac{x^3}{3} - 4x\right]_1^4$

$$= \left(\frac{5 \cdot 4^2}{2} - \frac{4^3}{3} - 4 \cdot 4\right) - \left(\frac{5 \cdot 1^2}{2} - \frac{1^3}{3} - 4 \cdot 1\right) = 9/2$$

7. a) $\frac{dy}{dx} = 4x^3 - 16x$

b) $\frac{dy}{dx} = 0 \Rightarrow 4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$4x(x-2)(x+2) = 0$

$x = 0$ $\underline{\underline{OR}}$ $x = 2$ $\underline{\underline{OR}}$ $x = -2$

$y = 3$ $\quad y = -13$ $\quad y = -13$

$(0,3)$ $\quad (2,-13)$ $\quad (-2,-13)$

c) $\frac{d^2y}{dx^2} = 12x^2 - 16$

$\left.\frac{d^2y}{dx^2}\right|_{x=0} = -16 < 0 \quad \therefore (0,3)$ is a MAXIMUM

$\left.\frac{d^2y}{dx^2}\right|_{x=2} = 32 > 0 \quad \therefore (2,-13)$ is a MINIMUM

$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = 32 > 0 \quad \therefore (-2,-13)$ is a MINIMUM

d) $\left.\frac{dy}{dx}\right|_{x=1} = 4 \cdot 1^3 - 16 \cdot 1 = -12 \Rightarrow m_{NORMAL} = \frac{1}{12}$

When $x=1$, $y = -4$ $\Rightarrow$ Equation of normal $y - (-4) = \frac{1}{12}(x-1)$

$12y + 48 = x - 1$

$12y - x + 49 = 0$

8. a) $9 - x = x^2 - 2x + 3$

$0 = x^2 - x - 6$

$0 = (x-3)(x+2)$

$x = 3$ $\underline{\underline{OR}}$ $x = -2$ $\quad \therefore A(-2,11) \quad B(3,6)$

$y = 6$ $\quad y = 11$

b) $R = \int_{-2}^3 (9-x) - (x^2 - 2x + 3)\,dx = \int_{-2}^3 -x^2 + x + 6\,dx = \left[-\frac{x^3}{3} + \frac{x^2}{2} + 6x\right]_{-2}^3$

$$= \left(-\frac{3^3}{3} + \frac{3^2}{2} + 6 \cdot 3\right) - \left(-\frac{(-2)^3}{3} + \frac{(-2)^2}{2} + 6(-2)\right) = 125/6$$

9. a) $5+2x-x^2=2$

$0=x^2-2x-3$

$0=(x-3)(x+1)$

$x=3$ OR $x=-1$
B A

b) $R=\int_{-1}^{3} 5+2x-x^2-2\,dx = \int_{-1}^{3} 3+2x-x^2\,dx = \left[3x+\frac{2x^2}{2}-\frac{x^3}{3}\right]_{-1}^{3}$

$= \left(3\cdot3+\frac{2\cdot3^2}{2}-\frac{3^3}{3}\right)-\left(3(-1)+\frac{2(-1)^2}{2}-\frac{(-1)^3}{3}\right) = 32/3$

10. a) $V=2x\cdot x\cdot h$

$1030=2x^2h$

$h=\frac{515}{x^2}$

b) $A=2(2x\cdot x)+2(2x\cdot h)+2(x\cdot h)$

$=4x^2+6xh = 4x^2+6x\cdot\frac{515}{x^2} = 4x^2+\frac{3090}{x}$ AS REQUIRED

c) $A=4x^2+3090x^{-1}$

$\frac{dA}{dx}=8x-3090x^{-2}$ Minimum $\Rightarrow \frac{dA}{dx}=0 \Rightarrow 8x-\frac{3090}{x^2}=0$

$8x=\frac{3090}{x^2}$

$8x^3=3090$

$x^3=386.25$

$x=\sqrt[3]{386.25}=7.28$

$A=4\left(\sqrt[3]{386.25}\right)^2+\frac{3090}{\sqrt[3]{386.25}} = 636\text{ cm}^2$ (to 3sf)

d) $\frac{d^2A}{dx^2}=8+6180x^{-3}$ $\left.\frac{d^2A}{dx^2}\right|_{x=\sqrt[3]{386.25}}=24>0$ $\therefore$ It is a minimum

11. a) $C=160v^{-1}+\frac{v^2}{100}$

$\frac{dC}{dv}=-160v^{-2}+\frac{2v}{100}$ Stationary $\Rightarrow \frac{dC}{dv}=0 \Rightarrow -160v^{-2}+\frac{2v}{100}=0$

$\frac{v}{50}=\frac{160}{v^2}$

$v^3=8000$

$v=20$

b) $\frac{d^2C}{dv^2}=320v^{-3}+\frac{2}{100}$ $\left.\frac{d^2C}{dv^2}\right|_{v=20}=3/50>0$ $\therefore$ This is a minimum

c) $C = \frac{160}{20} + \frac{20^2}{100} = 12$ $\Rightarrow$ Minimum cost for 250km journey $= 12 \times 250$
$= 3000$ pence $= £30$

12. a) $A = 2\pi r h + \pi r^2$
$250 = 2\pi r h + \pi r^2$

$h = \frac{250 - \pi r^2}{2\pi r}$

$V = \pi r^2 h = \pi r^2 \left(\frac{250 - \pi r^2}{2\pi r}\right) = \frac{r}{2}\left(250 - \pi r^2\right) = 125r - \frac{\pi r^3}{2}$ AS REQUIRED.

b) $\frac{dV}{dr} = 125 - \frac{3\pi r^2}{2}$ $\frac{dV}{dr} = 0 \Rightarrow 125 - \frac{3\pi r^2}{2} = 0$

$125 = \frac{3\pi r^2}{2}$

$r = \sqrt{\frac{250}{3\pi}} = 5.15$

c) $\frac{d^2V}{dr^2} = -\frac{6\pi r}{2}$ $\left.\frac{d^2V}{dr^2}\right|_{r=5.15} = -48.5 < 0$ $\therefore$ It is maximum

d) $V = 125\sqrt{\frac{250}{3\pi}} - \frac{3\pi}{2}\left(\sqrt{\frac{250}{3\pi}}\right)^2 = 519\ cm^3$ (to the nearest cm^3).

13 a) When $x = 4$, $y = 9 - 2(4) - \frac{2}{\sqrt{4}} = 0 \Rightarrow b = 4$ AS REQUIRED

b) $y = 9 - 2x - 2x^{-1/2}$ $\frac{dy}{dx} = -2 + x^{-3/2}$ $\left.\frac{dy}{dx}\right|_{x=1} = -2 + 1^{-3/2} = -1$

$\Rightarrow$ $y - 5 = -1(x - 1)$
$y - 5 = -x + 1$

$y + x = 6$ AS REQUIRED

c) At D, $y = 0$ $\Rightarrow x = 6$ $\therefore D(6,0)$

d) R = Area of triangle $- \int_1^4 9 - 2x - 2x^{-1/2}\,dx = \frac{5 \cdot 5}{2} - \left[9x - \frac{2x^2}{2} - \frac{2x^{1/2}}{1/2}\right]_1^4$

$= \frac{5 \cdot 5}{2} - \left[9x - x^2 - 4x^{1/2}\right]_1^4 = \frac{25}{2} - \left\{\left(9 \cdot 4 - 4^2 - 4 \cdot 4^{1/2}\right) - \left(9 \cdot 1 - 1^2 - 4 \cdot 1^{1/2}\right)\right\}$

$= 9/2$

14 a) At A $y=0 \Rightarrow \frac{3}{2}x^2 - \frac{1}{4}x^3 = 0$

$\Rightarrow x^2\left(\frac{3}{2} - \frac{1}{4}x\right) = 0$

$x = 0$ $\underline{\underline{OR}}$ $\frac{3}{2} = \frac{1}{4}x$

$x = 6 \Rightarrow p = 6$ AS REQUIRED

b) $\frac{dy}{dx} = 3x - \frac{3}{4}x^2$ $\left.\frac{dy}{dx}\right|_{x=6} = -9 \Rightarrow y - 0 = -9(x-6)$

$y = -9x + 54$

c) Maximum $\Rightarrow \frac{dy}{dx} = 0 \Rightarrow 3x - \frac{3}{4}x^2 = 0$

$3x\left(1 - \frac{1}{4}x\right) = 0$

$x = 0$ $\underline{\underline{OR}}$ $x = 4$

$\Rightarrow$ At P $x = 4$.

d) $R = \int_0^6 \frac{3}{2}x^2 - \frac{1}{4}x^3\,dx = \left[\frac{3}{2}\frac{x^3}{3} - \frac{1}{4}\frac{x^4}{4}\right]_0^6 = \left[\frac{x^3}{2} - \frac{x^4}{16}\right]_0^6 = \left(\frac{6^3}{2} - \frac{6^4}{16}\right) - 0 = 27$