C4 - Chapter 5 - Vectors - Extra practice - Solutions

1.a) $\overrightarrow{AB} = \begin{pmatrix} 8 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 5 \\ 8 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 9 \end{pmatrix}$

$$l: \underline{r} = \begin{pmatrix} 5 \\ 8 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -6 \\ 9 \end{pmatrix}$$

b) $\begin{pmatrix} 5 \\ 8 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -6 \\ 9 \end{pmatrix} = \begin{pmatrix} p \\ 4p \\ q \end{pmatrix} \Rightarrow 5 + 3\lambda = p$

$8 - 6\lambda = 4p$

Substitute $p = 5 + 3\lambda$

$8 - 6\lambda = 4(5 + 3\lambda)$

$8 - 6\lambda = 20 + 12\lambda$

$-12 = 18\lambda$

$\lambda = -2/3$

$\Rightarrow \boxed{p = 5 + 3\left(-2/3\right) = 3}$

$-4 + 9\lambda = q$

$q = -4 + 9\left(-2/3\right)$

$\boxed{q = -10}$

2.a) $\begin{pmatrix} 5+3t \\ 0-4t \\ 4+2t \end{pmatrix} = \begin{pmatrix} 5+2s \\ -1-3s \\ 9+3s \end{pmatrix}$

$5 + 3t = 5 + 2s$

$t = \frac{2}{3}s$

$0 - 4t = -1 - 3s$

Substitute $t = \frac{2}{3}s$

$-4\left(\frac{2}{3}s\right) = -1 - 3s$

$-\frac{8s}{3} = -1 - 3s$

$-8s = -3 - 9s$

$s = -3$

$t = -2$

$4 + 2t = 9 + 3s$

CHECK:

$4 + 2t = 4 + 2(-2) = 0$

$9 + 3s = 9 + 3(-3) = 0$

$\therefore$ The two lines intersect

b) Substitute $s = -3$ into l_2 (You could substitute $t = -2$ into l_1)

$\Rightarrow$ Point of intersection has coordinates $(-1, 8, 0)$

c) $\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix} = 6 + 12 + 6 = 24$

$|3\underline{i} - 4\underline{j} + 2\underline{k}| = \sqrt{3^2 + (-4)^2 + 2^2} = \sqrt{29}$

$|2\underline{i} - 3\underline{j} + 3\underline{k}| = \sqrt{2^2 + (-3)^2 + 3^2} = \sqrt{22}$

$\Rightarrow \cos\theta = \frac{24}{\sqrt{29}\sqrt{22}} \Rightarrow \theta = 18.2°$ (to 1dp)

3. a) $\vec{AB} = \begin{pmatrix}7\\1\\t\end{pmatrix} - \begin{pmatrix}3\\t\\5\end{pmatrix} = \begin{pmatrix}4\\1-t\\t-5\end{pmatrix}$

$|\vec{AB}| = \sqrt{4^2 + (1-t)^2 + (t-5)^2} = \sqrt{16 + 1 - 2t + t^2 + t^2 - 10t + 25}$

$= \sqrt{42 - 12t + 2t^2}$

b) Let $y = \sqrt{42 - 12t + 2t^2}$

To find the minimum, solve $\frac{dy}{dt} = 0$.

$$\frac{dy}{dt} = \frac{1}{2}(-12 + 4t)(42 - 12t + 2t^2)^{-1/2}$$

$$= \frac{1/2(-12+4t)}{\sqrt{42 - 12t + 2t^2}}$$

$$\frac{dy}{dt} = 0 \implies -12 + 4t = 0 \implies t = 3$$

c) Minimum $|\vec{AB}| = \sqrt{42 - 12(3) + 2(3)^2} = \sqrt{24} = 2\sqrt{6}$

4. a) $l_1 : \underset{\sim}{r} = \begin{pmatrix}1+\lambda\\3+2\lambda\\5-\lambda\end{pmatrix}$ $\qquad l_2 : \underset{\sim}{r} = \begin{pmatrix}-2+2\mu\\3+\mu\\-4+4\mu\end{pmatrix}$

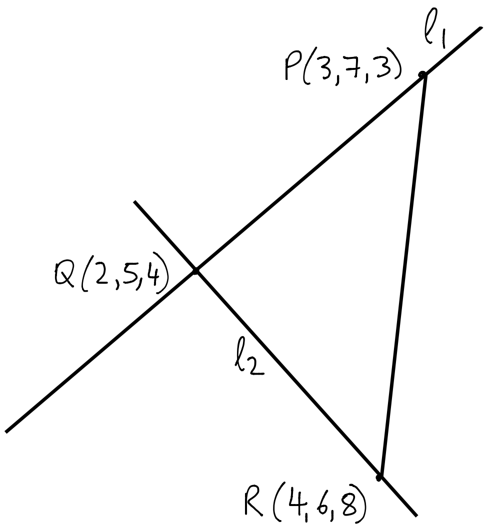
$1+\lambda = -2 + 2\mu$	$3+2\lambda = 3+\mu$	$5-\lambda = -4+4\mu$
$3+\lambda = 2\mu$	$2\lambda = \mu$	CHECK
Substitute $\mu = 2\lambda$	$\mu = 2$	$5-\lambda = 5-1 = 4$
$\implies 3+\lambda = 4\lambda$		$-4+4\mu = -4+4(2) = 4$
$3\lambda = 3 \implies \lambda = 1$		$\therefore$ The two lines intersect

To find the coordinates of Q, substitute $\lambda = 1$ into $l_1 \implies Q(2, 5, 4)$

b) $\begin{pmatrix}1\\2\\-1\end{pmatrix} \cdot \begin{pmatrix}2\\1\\4\end{pmatrix} = 2 + 2 - 4 = 0$ $\quad \therefore l_1$ and l_2 are perpendicular AS REQUIRED

c) Let P have coordinates $(3, y_1, z_1)$ and R have coordinates $(4, y_2, z_2)$.

Then, $\begin{pmatrix}1+\lambda\\3+2\lambda\\5-\lambda\end{pmatrix} = \begin{pmatrix}3\\y_1\\z_1\end{pmatrix}$ $\quad 1+\lambda = 3 \implies \lambda = 2 \implies y_1 = 7,\ z_1 = 3$ $\qquad \begin{pmatrix}-2+2\mu\\3+\mu\\-4+4\mu\end{pmatrix} = \begin{pmatrix}4\\y_2\\z_2\end{pmatrix}$ $\quad -2+2\mu = 4 \implies \mu = 3 \implies y_2 = 6,\ z_2 = 8$



$PQ = \sqrt{(3-2)^2 + (7-5)^2 + (3-4)^2} = \sqrt{6}$

$QR = \sqrt{(4-2)^2 + (6-5)^2 + (8-4)^2} = \sqrt{21}$

$\Rightarrow$ Area of $\triangle PQR = \dfrac{\sqrt{6}\ \sqrt{21}}{2} = \dfrac{3\sqrt{14}}{2}$

5 a) $\vec{BA} = \begin{pmatrix}3\\2\\-1\end{pmatrix} - \begin{pmatrix}5\\1\\1\end{pmatrix} = \begin{pmatrix}-2\\1\\-2\end{pmatrix}$

$\vec{BC} = \begin{pmatrix}7\\-1\\0\end{pmatrix} - \begin{pmatrix}5\\1\\1\end{pmatrix} = \begin{pmatrix}2\\-2\\-1\end{pmatrix}$

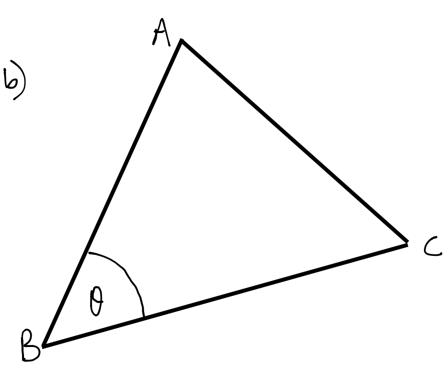
$\vec{BA} \cdot \vec{BC} = \begin{pmatrix}-2\\1\\-2\end{pmatrix} \cdot \begin{pmatrix}2\\-2\\-1\end{pmatrix} = -4-2+2 = -4$

$|\vec{BA}| = \sqrt{(-2)^2 + 1^2 + (-2)^2} = 3$

$|\vec{BC}| = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3$

$\therefore \cos(A\hat{B}C) = \dfrac{-4}{9}$

b)



Area of $\triangle ABC = \frac{1}{2} 3.3 \sin\theta$

$= \frac{1}{2}\ 9\ \sqrt{1-\cos^2\theta}$

$= \frac{9}{2} \sqrt{1 - \left(\frac{-4}{9}\right)^2} = \dfrac{\sqrt{65}}{2}$

c) $\overrightarrow{AC} = \begin{pmatrix} 7 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$

$\overrightarrow{CD} = \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 7 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$

$\overrightarrow{AC} \cdot \overrightarrow{CD} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = -3+3=0$

$\therefore$ AC is perpendicular to CD AS REQUIRED

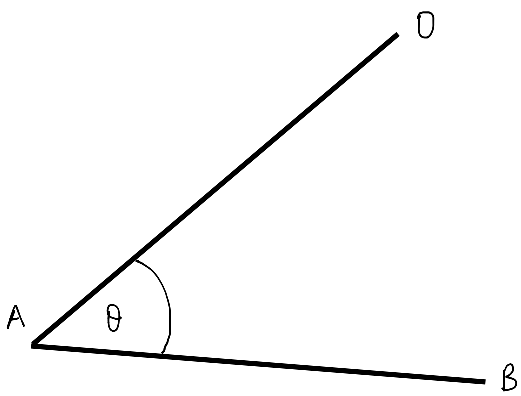
d) $AD = \sqrt{(7-3)^2 + (0-2)^2 + [3-(-1)]^2} = 6$

$DB = \sqrt{(7-5)^2 + (0-1)^2 + (3-1)^2} = 3$

$AD : DB = 2 : 1$

6.a) $\overrightarrow{AB} = \begin{pmatrix} 7 \\ 14 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 8 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix}$

b)



$\overrightarrow{AO} \cdot \overrightarrow{AB} = |\overrightarrow{AO}| \, |\overrightarrow{AB}| \cos\theta$

$\begin{pmatrix} -4 \\ -8 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} = -12-48+6 = -54$

$|-4\underset{\sim}{i} - 8\underset{\sim}{j} + \underset{\sim}{k}| = \sqrt{(-4)^2 + (-8)^2 + 1^2} = 9$

$|3\underset{\sim}{i} + 6\underset{\sim}{j} + 6\underset{\sim}{k}| = \sqrt{3^2+6^2+6^2} = 9$

$\Rightarrow \cos\theta = \frac{-54}{9\times 9} = -2/3$

c) AB has equation $\begin{pmatrix} 4 \\ 8 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 4+3\mu \\ 8+6\mu \\ -1+6\mu \end{pmatrix}$

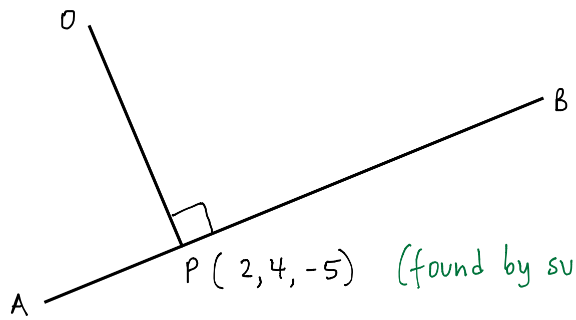
$\begin{pmatrix} 4+3\mu \\ 8+6\mu \\ -1+6\mu \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda \\ 2\lambda-9 \end{pmatrix}$ $\quad \lambda = 4+3\mu \quad$ $2\lambda = 8+6\mu$, $\lambda = 4+3\mu \quad$ $-1+6\mu = 2\lambda - 9$, $8+6\mu = 2\lambda$, $\lambda = 4+3\mu$

$\therefore$ P lies on the line through A and B AS REQUIRED

d) $\vec{OP} \cdot \vec{AB} = \begin{pmatrix} \lambda \\ 2\lambda \\ 2\lambda - 9 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} = 3\lambda + 12\lambda + 12\lambda - 54 = 0 \quad \Rightarrow \quad 27\lambda - 54 = 0$

$\lambda = 2$

e)



P (2, 4, -5) (found by substituting $\lambda = 2$)