CORE MATHEMATICS 4
CHAPTER 6 – INTEGRATION
EXTRA PRACTICE

1.

$$\frac{dy}{dx} = \frac{2xy}{1+x^2},\ y > 0.$$

(*a*) Find the general solution of this differential equation.

Given that $y = 2$ at $x = 0$,

(*b*) find the particular solution in the form $y = f(x)$.

2.

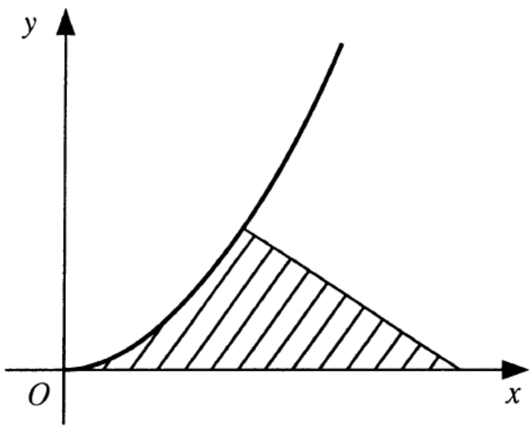


Fig. 1

The curve shown in Fig. 1 has parametric equations

$$x = t^2,\ y = t^3,$$

where $t \geqslant 0$ is a parameter. Also shown is part of the normal to the curve at the point where $t = 1$.

(*a*) Find an equation of this normal.

(*b*) Find the area of the finite region bounded by the curve, the x-axis and this normal.

---

3. Use integration by parts to show that

$$\int_2^4 x \ln x \, dx = 7 \ln 4 - 3.$$

---

4. A bottle is shaped so that when the depth of water is x cm, the volume of water in the bottle is $(x^2 + 4x)$ cm^3, $x \geqslant 0$. Water is poured into the bottle so that at time t s after pouring commences, the depth of water is x cm and the rate of increase of the volume of the water is $(x^2 - 25)$ cm^3 s^{-1}.

(*a*) Show that $\dfrac{dx}{dt} = \dfrac{x^2 - 25}{2x + 4}$.

Given that the bottle was empty at $t = 0$,

(*b*) solve this differential equation to obtain t in terms of x.

---

5. (*a*) (i) Expand $(\cos\theta + \sin\theta)^2$ and simplify the result.

(ii) Show that $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos\theta + \sin\theta)^2 \, d\theta = \frac{\pi}{4} + \frac{1}{2}$

(*b*) Work out $\displaystyle\int_0^1 (2x + 1)^4 \, dx$.

---

6. Express as the sum of partial fractions

$$\frac{2}{x(x + 1)(x + 2)}.$$

Hence show that

$$\int_2^4 \frac{2}{x(x + 1)(x + 2)} \, dx = 3 \ln 3 - 2 \ln 5.$$

---

7. (*a*) Given that $2y = x - \sin x \cos x$, show that $\dfrac{dy}{dx} = \sin^2 x$.

(*b*) Hence find $\displaystyle\int \sin^2 x \, dx$.

(*c*) Hence, using integration by parts, find $\displaystyle\int x \sin^2 x \, dx$.

---

8. At time t hours the rate of decay of the mass of a radioactive substance is proportional to the mass x kg of the substance at that time. At time $t = 0$ the mass of the substance is A kg.

(*a*) By forming and integrating a differential equation, show that $x = Ae^{-kt}$, where k is a constant.

It is observed that $x = \frac{1}{3}A$ at time $t = 10$.

(*b*) Find the value of t when $x = \frac{1}{2}A$, giving your answer to 2 decimal places.

---

9. Find

(*a*) $\int \sin 2y \, dy$,

(*b*) $\int xe^x dx$.

(*c*) Hence find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{xe^x}{\sin y \cos y}, \qquad 0 < y < \frac{\pi}{2}.$$

---

10. $$f(x) \equiv \frac{5x^2 - 8x + 1}{2x(x-1)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

(*a*) Find the values of the constants A, B and C.

(*b*) Hence find $\int f(x) \, dx$.

(*c*) Hence show that

$$\int_4^9 f(x) \, dx = \ln\left(\frac{32}{3}\right) - \frac{5}{24}.$$

---

11. (*a*) Find $\int x \sin 2x \, dx$.

(*b*) Given that $y = 0$ at $x = \frac{\pi}{4}$, solve the differential equation

$$\frac{dy}{dx} = x \sin 2x \cos^2 y.$$

---

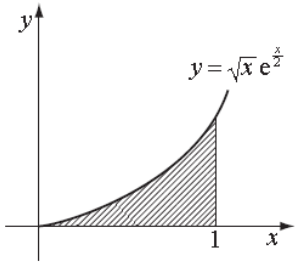
12. (*a*) Find the exact value of $\int_1^2 \frac{2x^2+1}{x}\, dx$.

(*b*) (i) Use integration by parts to show that

$$\int x e^x \, dx = e^x(x-1) + c.$$

(ii) The sketch shows the graph of

$$y = \sqrt{x}\, e^{\frac{x}{2}}.$$



The region R, enclosed by the curve and the lines $y = 0$ and $x = 1$, is rotated through four right angles about the x-axis. Find the exact volume of the solid formed.

---

13. (*a*) Express

$$\frac{8x+1}{(2x-1)(x+2)}$$

in partial fractions.

(*b*) Solve the differential equation

$$\frac{dy}{dx} = \frac{(8x+1)y}{(2x-1)(x+2)},$$

given that $y = 1$ when $x = 1$. Give your answer in the form $y = f(x)$.

14.

Figure 1

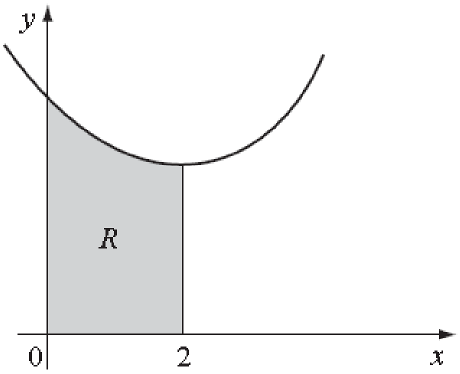


Figure 1 shows a sketch of the curve with parametric equations

$$x = t + 2, \qquad y = t^2 + 1$$

The region R is bounded by the curve and the lines $y = 0, x = 0$ and $x = 2$.

When R is rotated through 360° about the x-axis the volume generated is V.

(a) Show that $V = \pi \int_{t=-2}^{t=0} \left(t^2 + 1\right)^2 \, \mathrm{d}t$

(b) Find the exact value of V.

---

15. (a) Use the trapezium rule, with the number of trapeziums given below, to find approximate values for

$$\int_1^3 \frac{12}{x} \mathrm{d}x,$$

(i) using two trapeziums,

(ii) using four trapeziums.

(b) Work out the exact value for this integral. For each case in part (a) work out the error when using the trapezium rule as a percentage of the exact value, giving your answers correct to one decimal place.
