C4 - Chapter 6 - Integration - Extra practice 2 - Solutions

1. a)

$$\frac{dy}{dx} = \frac{2xy}{1+x^2}$$

$$\int \frac{1}{y}\,dy = \int \frac{2x}{1+x^2}\,dx \qquad \text{Let } u = 1+x^2, \quad du = 2x\,dx$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{u}\,du$$

$$\ln|y| = \ln|u| + c$$

$$\ln|y| = \ln|1+x^2| + c$$

b) When $x=0, y=2 \Rightarrow \ln 2 = \ln 1 + c \Rightarrow c = \ln 2$

$$\ln|y| = \ln|1+x^2| + \ln 2$$

$$\ln|y| = \ln|2(1+x^2)|$$

$$y = 2(1+x^2)$$

2 a) $x = t^2 \quad y = t^3$

$\frac{dx}{dt} = 2t \quad \frac{dy}{dt} = 3t^2 \quad \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2}{2t} = \frac{3t}{2}$

When $t=1$ $x=1, y=1, \frac{dy}{dx} = \frac{3}{2} \Rightarrow m_{NORMAL} = -2/3$

$$\therefore \; y-1 = -\frac{2}{3}(x-1)$$

b) Area $= \int_0^1 y\,dx$ + Area of triangle

Now, normal will cut the x-axis at $x = 5/2$ (found by setting $y=0$)

$\therefore$ Triangle has height 1 and base $5/2 - 1 = 3/2$

$$\Rightarrow \text{Area} = \int_0^1 t^3 \cdot 2t\,dt + \frac{1\times 3/2}{2} = \int_0^1 2t^4\,dt + \frac{3}{4} = \left[\frac{2t^5}{5}\right]_0^1 + \frac{3}{4}$$

$$= 2/5 + 3/4 = 23/20$$

3. $$\int_2^4 x\ln x\,dx = \left[\frac{x^2}{2}\ln x\right]_2^4 - \int_2^4 \frac{1}{x}\cdot\frac{x^2}{2}\,dx = \left(\frac{4^2}{2}\ln 4 - \frac{2^2}{2}\ln 2\right) - \int_2^4 \frac{x}{2}\,dx$$

$u = \ln x \quad \frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x} \quad v = \frac{x^2}{2}$

$$= 8\ln 4 - 2\ln 2 - \left[\frac{x^2}{4}\right]_2^4$$

$$= 8\ln 4 - \ln 2^2 - \left(\frac{4^2}{4} - \frac{2^2}{4}\right)$$

$$= 8\ln 4 - \ln 4 - 3 = 7\ln 4 - 3 \quad \text{AS REQUIRED}$$

4. a) $V = x^2 + 4x \qquad \frac{dV}{dt} = x^2 - 25$

$\frac{dV}{dx} = 2x + 4$

$$\begin{tikzcd} & V \arrow[dl, <->] \arrow[dr, <->] & \\ x \arrow[rr, <->, dashed] & & t \end{tikzcd}$$

$\frac{dx}{dt} = \frac{dx}{dV} \cdot \frac{dV}{dt} = \frac{1}{2x+4} \cdot x^2 - 25 = \frac{x^2-25}{2x+4}$ AS REQUIRED

b) $\int \frac{2x+4}{x^2-25}\, dx = \int dt$

$\frac{2x+4}{x^2-25} = \frac{2x+4}{(x-5)(x+5)} = \frac{A}{x-5} + \frac{B}{x+5}$

$2x+4 = A(x+5) + B(x-5)$

$x = 5 \Rightarrow 14 = 10A \Rightarrow A = 7/5$

$x = -5 \Rightarrow -6 = -10B \Rightarrow B = 3/5$

$\int \frac{7/5}{x-5} + \frac{3/5}{x+5}\, dx = \int dt$

$\frac{7}{5} \ln|x-5| + \frac{3}{5} \ln|x+5| = t + c$

When $t=0$, $x=0$ (since bottle was empty, hence height was 0)

$\Rightarrow \frac{7}{5} \ln|-5| + \frac{3}{5} \ln 5 = 0 + c$

$c = 2 \ln 5$

$\Rightarrow \frac{7}{5} \ln|x-5| + \frac{3}{5} \ln|x+5| = t + 2\ln 5$

$\Rightarrow t = \frac{7}{5} \ln|x-5| + \frac{3}{5} \ln|x+5| - 2\ln 5$

5a) i) $(\cos\theta + \sin\theta)^2 = \cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta$

$= 1 + \sin 2\theta \qquad (\text{since } \sin^2\theta + \cos^2\theta = 1)$

ii) $\int_{\pi/4}^{\pi/2} (\cos\theta + \sin\theta)^2\, d\theta = \int_{\pi/4}^{\pi/2} (1 + \sin 2\theta)\, d\theta = \left[\theta - \frac{1}{2}\cos 2\theta\right]_{\pi/4}^{\pi/2}$

$= \left(\pi/2 - \frac{1}{2}\cos\pi\right) - \left(\pi/4 - \frac{1}{2}\cos(\pi/2)\right)$

$= \pi/4 + 1/2$ AS REQUIRED

b) $\int_0^1 (2x+1)^4\, dx = \left[\frac{(2x+1)^5}{5 \cdot 2}\right]_0^1 = \frac{3^5}{10} - \frac{1^5}{10} = 24.2$

6. $\dfrac{2}{x(x+1)(x+2)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x+2} \Rightarrow 2 = A(x+1)(x+2) + Bx(x+2) + Cx(x+1)$

$x=0 \Rightarrow 2=2A \Rightarrow A=1$

$x=-1 \Rightarrow 2=-B \Rightarrow B=-2$

$x=-2 \Rightarrow 2=2C \Rightarrow C=1$

$\therefore \dfrac{2}{x(x+1)(x+2)} = \dfrac{1}{x} - \dfrac{2}{x+1} + \dfrac{1}{x+2}$

$$\int_2^4 \frac{2}{x(x+1)(x+2)}\,dx = \int_2^4 \frac{1}{x} - \frac{2}{x+1} + \frac{1}{x+2}\,dx = \Big[\ln x - 2\ln x+1 + \ln x+2\Big]_2^4$$

$$= \Big[\ln x - \ln(x+1)^2 + \ln x+2\Big]_2^4 = \left[\ln \frac{x(x+2)}{(x+1)^2}\right]_2^4$$

$$= \ln\left(\frac{4(4+2)}{(4+1)^2}\right) - \ln\left(\frac{2(2+2)}{(2+1)^2}\right)$$

$$= \ln\left(\frac{24}{25}\right) - \ln\left(\frac{8}{9}\right) = \ln\left(\frac{24/25}{8/9}\right) = \ln\frac{27}{25}$$

$$= \ln 27 - \ln 25 = \ln 3^3 - \ln 5^2 = 3\ln 3 - 2\ln 5$$ AS REQUIRED

7 a) $2y = x - \sin x\cos x$

$2\dfrac{dy}{dx} = 1 - \left(\cos^2 x - \sin^2 x\right)$ Implicit differentiation and product rule

$2\dfrac{dy}{dx} = 1 - \cos^2 x + \sin^2 x$

$2\dfrac{dy}{dx} = \sin^2 x + \sin^2 x \Rightarrow \dfrac{dy}{dx} = \sin^2 x$ AS REQUIRED

b) $\displaystyle\int \sin^2 x\,dx = \frac{x - \sin x\cos x}{2} + C$

OK, b) is difficult. From a) we concluded that if $2y = x - \sin x\cos x$ (*)

then $\dfrac{dy}{dx} = \sin^2 x \Rightarrow \int dy = \int \sin^2 x\,dx$

$\Rightarrow y = \int \sin^2 x\,dx = \dfrac{x - \sin x\cos x}{2} + C$ using (*)

c) $\displaystyle\int x\sin^2 x\,dx = x\left(\frac{x-\sin x\cos x}{2}\right) - \int \frac{x - \sin x\cos x}{2}\,dx = x\left(\frac{x-\sin x\cos x}{2}\right) - \int \frac{x}{2} - \frac{1}{4}\sin 2x\,dx$

$u = x \qquad \dfrac{dv}{dx} = \sin^2 x$

$$= x\left(\frac{x - \sin x\cos x}{2}\right) - \left(\frac{x^2}{4} + \frac{1}{8}\cos 2x\right) + C$$

$\dfrac{du}{dx} = 1 \qquad v = \dfrac{x - \sin x\cos x}{2}$

8. a) $\frac{dx}{dt} = -kx \Rightarrow \int \frac{1}{x}\,dx = \int -k\,dt$

$$\ln|x| = -kt + C$$

When $t=0$, $x=A \Rightarrow C = \ln A$

$$\ln|x| = -kt + \ln A$$

$$x = e^{-kt+\ln A} = e^{-kt} \cdot e^{\ln A}$$

$\Rightarrow x = Ae^{-kt}$ AS REQUIRED

b) When $t=10$, $x = \frac{1}{3}A \Rightarrow \frac{1}{3}A = Ae^{-10k}$

$$1/3 = e^{-10k}$$

$$\ln(1/3) = -10k$$

$$k = -\frac{1}{10}\ln(1/3) = \frac{1}{10}\ln 3$$

When $x = \frac{1}{2}A$, $\frac{1}{2}A = Ae^{-t \cdot \frac{1}{10}\ln 3}$

$$\frac{1}{2} = e^{-t\frac{1}{10}\ln 3}$$

$$\ln(1/2) = -t \cdot \frac{1}{10}\ln 3$$

$$t = \frac{-\ln(1/2)}{\frac{1}{10}\ln 3} = 6.31 \quad \text{(to 2dp's)}$$

9. a) $\int \sin 2y\,dy = -\frac{1}{2}\cos 2y + C$

b) $\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$

$u = x \qquad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1 \qquad v = e^x$

c) $$\frac{dy}{dx} = \frac{xe^x}{\sin y \cos y}$$

$$\int \sin y \cos y\,dy = \int xe^x\,dx$$

$$\int \frac{1}{2}\sin 2y\,dy = \int xe^x\,dx$$

$$-\frac{1}{4}\cos 2y = xe^x - e^x + C$$

10. a) $\dfrac{5x^2-8x+1}{2x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$

Careful!

$5x^2-8x+1 = 2A(x-1)^2 + 2Bx(x-1) + 2Cx$

$x=0 \Rightarrow 1 = 2A \Rightarrow A = 1/2$

$x=1 \Rightarrow -2 = 2C \Rightarrow C = -1$

$x=-1 \Rightarrow 14 = 8A + 4B - 2C$

$14 = 4 + 4B + 2$

$8 = 4B \Rightarrow B = 2$

$\therefore \dfrac{5x^2-8x+1}{2x(x-1)^2} = \dfrac{1/2}{x} + \dfrac{2}{x-1} - \dfrac{1}{(x-1)^2}$

b) $\int f(x)\,dx = \int \dfrac{1/2}{x} + \dfrac{2}{x-1} - \dfrac{1}{(x-1)^2}\,dx = \int \dfrac{1/2}{x} + \dfrac{2}{x-1} - (x-1)^{-2}\,dx$

$= \frac{1}{2}\ln|x| + 2\ln|x-1| - \dfrac{(x-1)^{-1}}{-1} + C$

c) $\int_4^9 f(x)\,dx = \left[\frac{1}{2}\ln x + 2\ln|x-1| + \frac{1}{x-1}\right]_4^9$

$= \left(\frac{1}{2}\ln 9 + 2\ln 8 + \frac{1}{8}\right) - \left(\frac{1}{2}\ln 4 + 2\ln 3 + \frac{1}{3}\right)$

$= \ln 3 + 6\ln 2 + \frac{1}{8} - \ln 2 - 2\ln 3 - \frac{1}{3} = 5\ln 2 - \ln 3 - \frac{5}{24}$

$= \ln 32 - \ln 3 - \frac{5}{24} = \ln\left(\frac{32}{3}\right) - \frac{5}{24}$ AS REQUIRED

11. a) $\int x\sin 2x\,dx = -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x\,dx = -\frac{1}{2}x\cos 2x + \int \frac{1}{2}\cos 2x\,dx$

$u = x \qquad \frac{dv}{dx} = \sin 2x$

$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$

$\frac{du}{dx} = 1 \qquad v = -\frac{1}{2}\cos 2x$

b) $\frac{dy}{dx} = x\sin 2x\cos^2 y$

$\int \frac{1}{\cos^2 y}\,dy = \int x\sin 2x\,dx$

$\int \sec^2 y\,dy = \int x\sin 2x\,dx$

$\tan y = \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$

When $x = \pi/4$, $y = 0 \Rightarrow \tan 0 = \frac{1}{2}\,\frac{\pi}{4}\cos\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin\left(\frac{\pi}{2}\right) + c \Rightarrow 0 = \frac{1}{4} + C \Rightarrow C = -\frac{1}{4}$

$\tan y = \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x - \frac{1}{4}$

12 a) $\int_1^2 \frac{2x^2+1}{x}\,dx = \int_1^2 \frac{2x^2}{x} + \frac{1}{x}\,dx = \int_1^2 2x + \frac{1}{x}\,dx = \left[\frac{2x^2}{2} + \ln x\right]_1^2$

$= (2^2 + \ln 2) - (1^2 + \ln 1) = 3 + \ln 2$

b) i) $\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C = e^x(x-1) + C$ AS REQUIRED

$u = x \qquad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1 \qquad v = e^x$

ii) Volume of revolution $= \pi\int_0^1 y^2\,dx = \pi\int_0^1 \left(\sqrt{x}\,e^{x/2}\right)^2 dx = \pi\int_0^1 xe^x\,dx\ dx$

$= \pi\left[e^x(x-1)\right]_0^1 = \pi\left\{\left(e^1\cancel{(1-1)}^{\,0}\right) - \left(e^0(0-1)\right)\right\}$

$= \pi$

13 a) $\frac{8x+1}{(2x-1)(x+2)} = \frac{A}{2x-1} + \frac{B}{x+2}$

$8x+1 = A(x+2) + B(2x-1)$

$x=-2 \Rightarrow -15 = -5B \Rightarrow B=3$

$x=\frac{1}{2} \Rightarrow 5 = \frac{5}{2}A \Rightarrow A=2$

$\therefore \frac{8x+1}{(2x-1)(x+2)} = \frac{2}{2x-1} + \frac{3}{x+2}$

b) $\frac{dy}{dx} = \frac{(8x+1)y}{(2x-1)(x+2)} \Rightarrow \int \frac{1}{y}\,dy = \int \frac{8x+1}{(2x-1)(x+2)}\,dx$

$\int \frac{1}{y}\,dy = \int \frac{2}{2x-1} + \frac{3}{x+2}\,dx$

$\ln|y| = \frac{2}{2}\ln|2x-1| + 3\ln|x+2| + C$

When $x=1, y=1 \Rightarrow \ln 1 = \ln 1 + 3\ln 3 + C \Rightarrow C = -3\ln 3$

$\therefore \ln|y| = \frac{2}{2}\ln|2x-1| + 3\ln|x+2| - 3\ln 3$

$\ln y = \ln(2x-1) + \ln(x+2)^3 - \ln 27$

$\ln y = \ln\left(\frac{(2x-1)(x+2)^3}{27}\right)$

$y = \frac{(2x-1)(x+2)^3}{27}$

14 a) $x = t+2 \qquad y = t^2+1$

$dx = dt$

When $x=0$, $0 = t+2 \Rightarrow t=-2$

When $x=2$, $2 = t+2 \Rightarrow t=0$

$$V = \pi \int_{x=0}^{x=2} y^2\,dx = \pi \int_{t=-2}^{t=0} (t^2+1)^2\,dt \quad \text{AS REQUIRED}$$

b) $$V = \pi \int_{-2}^{0} t^4+2t^2+1\,dt = \pi \left[\frac{t^5}{5} + \frac{2t^3}{3} + t\right]_{-2}^{0} = \pi \left\{ 0 - \left(\frac{(-2)^5}{5} + \frac{2(-2)^3}{3} + (-2) \right) \right\} = \frac{206\pi}{15}$$

15 a) i)

x	1	2	3
$y = 12/x$	12	6	4

$$\int_1^3 \frac{12}{x}\,dx \approx \frac{1}{2}\cdot 1 \left\{ 12 + 2(6) + 4 \right\} = 14$$

ii)

x	1	1.5	2	2.5	3
$y = 12/x$	12	8	6	24/5	4

$$\int_1^3 \frac{12}{x}\,dx \approx \frac{1}{2}\cdot 0.5 \left\{ 12 + 2\left(8+6+\frac{24}{5}\right) + 4 \right\} = \frac{67}{5}$$

b) $$\int_1^3 \frac{12}{x}\,dx = \left[12\ln x\right]_1^3 = 12\ln 3 - 12\ln 1 = 12\ln 3$$

i) $$\%\text{ error} = \frac{|14 - 12\ln 3|}{12\ln 3} \times 100\% = 6.2\%$$

ii) $$\%\text{ error} = \frac{|67/5 - 12\ln 3|}{12\ln 3} \times 100\% = 1.6\%$$