C4 - Chapter 6 - Integration - Summary

Integration is NOT hard, integration is LONG.

You should first familiarise yourself with the following standard integrals

1. $\int (ax+b)^n \, dx = \frac{1}{a} \frac{(ax+b)^{n+1}}{n+1} + C$, $n \neq -1$
2. $\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln|ax+b| + C$
3. $\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$
4. $\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + C$
5. $\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + C$
6. $\int \sec^2(ax+b) = \frac{1}{a} \tan(ax+b) + C$
7. $\int \operatorname{cosec}^2(ax+b) \, dx = \frac{-1}{a} \cot(ax+b) + C$
8. $\int \sec(ax+b) \tan(ax+b) \, dx = \frac{1}{a} \sec(ax+b) + C$
9. $\int \operatorname{cosec}(ax+b) \cot(ax+b) \, dx = \frac{-1}{a} \operatorname{cosec}(ax+b) + C$

Results ①-⑨ are valid only when a linear function of x is involved

LINEAR	NOT LINEAR
$2x+1$	$2x^2+1$
$-3x$	$-3\sqrt{x}$
e^{5x-2}	e^{5x^4-2}
$\ln\|7x-23\|$	$\ln\|7x^{1/3}-23\|$
$\tan(-x-1)$	$\tan(-\ln x - 1)$
$\cos(2-x)$	$\cos\left(2-\frac{1}{x}\right)$
$\sin(3x-4)$	$\sin(3e^x-4)$

* Don't forget the constant of integration (ie $+C$) when you are working with indefinite integrals (ie without limits). Omitting it can be costly!
* Notice that the only two products that have standard integrals are

  $\sec x \tan x$ and $\operatorname{cosec} x \cot x$

  So, please avoid things like $\int x^2 \cos x \, dx = \frac{x^3}{3} \sin x + C$ ✗

The results above are the basic weapons you have available to "attack" any question on Integration. Unfortunately, it is quite unlikely you will be given a standard integral to integrate. You will therefore have to be creative (!) in using techniques you already know (eg trigonometric identities) in order to express the given integral in a form which you know how to integrate.

Please bear in mind the following strategies or approaches as they are usually (but not always) a good way of simplifying the integral:

- Prefer to express fractions with a single term in the denominator rather than two or more terms. This allows us to split fractions as shown in the example below:

$$\int \frac{x^2+3x+5}{x}\,dx = \int \frac{x^2}{x} + \frac{3x}{x} + \frac{5}{x}\,dx = \int x+3+\frac{5}{x}\,dx = \frac{x^2}{2} + 3x + 5\ln|x| + C$$

$$\int \frac{\cos x}{1-\cos^2 x}\,dx = \int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}\,dx = \int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$$

- With integrals that involve trigonometric ratios try to express powers in terms of multiples of angles

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}x - \frac{\sin 2x}{4} + C \qquad \left(\text{using } \cos 2x = 1-2\sin^2 x\right)$$

- When integrating using substitution, it is usually better to put u equal to the complicated part of the integral

$$\int (x^2+2x)(x^3+3x^2+5)^{11}\,dx$$

$$= \int \frac{1}{3} u^{11}\,du = \frac{1}{3}\frac{u^{12}}{12} + C$$

$$= \frac{1}{36}(x^3+3x^2+5)^{12} + C$$

$$u = x^3+3x^2+5$$

$$\frac{du}{dx} = 3x^2+6x = 3(x^2+2x)$$

$$\Rightarrow \quad \frac{1}{3}\,du = (x^2+2x)\,dx$$

- Use trigonometric identities to express the integral in a form that can be integrated. Such trigonometric identities include:

$\sin^2 x + \cos^2 x = 1$ $\quad$ $\tan x = \frac{\sin x}{\cos x}$ $\quad$ $\sin 2x = 2\sin x\cos x$

$\sec x = \frac{1}{\cos x}$ $\quad$ $1 + \tan^2 x = \sec^2 x$ $\quad$ $\cos 2x = 2\cos^2 x - 1$

$\operatorname{cosec} x = \frac{1}{\sin x}$ $\quad$ $1 + \cot^2 x = \operatorname{cosec}^2 x$ $\quad$ $= 1 - 2\sin^2 x$

$= \cos^2 x - \sin^2 x$

$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$ $\quad$ Sum to product formulae (formula booklet)

Please note that these identities hold for any x

e.g. $\sin^2 2x + \cos^2 2x = 1$ $\quad$ $\sin 6x = 2\sin 3x \cos 3x$

$1 + \tan^2 4x = \sec^2 4x$ $\quad$ $\cos 8x = 2\cos^2 4x - 1$

Now that we know all the standard integrals, let's see some other techniques

① Partial fractions : First check if the fraction is improper. If yes, do long division

Factorise the denominator - check for repeated factors

Express in partial fractions

e.g. $\int \frac{x^4 - 5x + 2}{x^2 - x}\, dx = I$

$$\begin{array}{rl}
 & x^2 + x + 1 \\
x^2 - x \,\big) & x^4 + 0x^3 + 0x^2 - 5x + 2 \\
- & \underline{x^4 - x^3} \\
 & x^3 + 0x^2 - 5x + 2 \\
- & \underline{x^3 - x^2} \\
 & x^2 - 5x + 2 \\
- & \underline{x^2 - x} \\
 & -4x + 2
\end{array}$$

$$\therefore \frac{x^4 - 5x + 2}{x^2 - x} = x^2 + x + 1 + \frac{-4x + 2}{x(x-1)}$$

$$\frac{-4x+2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow -4x + 2 = A(x-1) + Bx$$

$x = 1 \Rightarrow B = -2 \qquad x = 0 \Rightarrow A = -2$

$$\therefore \frac{x^4 - 5x + 2}{x^2 - x} = x^2 + x + 1 - \frac{2}{x} - \frac{2}{x-1}$$

$$I = \int x^2 + x + 1 - \frac{2}{x} - \frac{2}{x-1}\, dx = \frac{x^3}{3} + \frac{x^2}{2} + x - 2\ln|x| - 2\ln|x-1| + C$$

② Substitution: Introduce a new variable, u say, and "translate" the original integral from in terms of x to in terms of u, in an attempt to make it simpler.

This means that you need to: * Change all terms in x to u

* express dx in terms of du

* change the limits (if there are any) from x to u

Remember to "translate" your answer back from u to x.

e.g. Use the substitution $u = 5x+3$, to find the exact value of

$$I = \int_0^3 \frac{10x}{(5x+3)^3}\,dx$$

$u = 5x+3$ $\qquad$ $x = \frac{u-3}{5}$ $\qquad$ When $x=0$, $u=3$

$du = 5\,dx$ $\qquad$ When $x=3$, $u=18$

$$\therefore I = \int_0^3 \frac{2x \cdot 5}{(5x+3)^3}\,dx = \int_3^{18} 2\left(\frac{u-3}{5}\right)\frac{1}{u^3}\,du$$

$$= \frac{2}{5}\int_3^{18} \frac{u-3}{u^3}\,du = \frac{2}{5}\int_3 \frac{u}{u^3} - \frac{3}{u^3}\,du$$

$$= \frac{2}{5}\int_3^{18} \left(u^{-2} - 3u^{-3}\right)du$$

$$= \frac{2}{5}\left[\frac{u^{-1}}{-1} - \frac{3u^{-2}}{-2}\right]_3^{18} = \frac{2}{5}\left[\frac{-1}{u} + \frac{3}{2u^2}\right]_3^{18}$$

$$= \frac{2}{5}\left\{\left(\frac{-1}{18} + \frac{3}{2\cdot 18^2}\right) - \left(\frac{-1}{3} + \frac{3}{2\cdot 3^2}\right)\right\}$$

$$= \frac{5}{108}$$

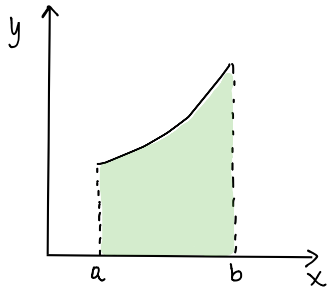
③ Integration by parts: Use this to integrate products when all other techniques fail

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

In general, put u to be that part that when differentiated will result in a much simpler integral.

SPECIAL CASE: If $\ln x$ is present then that should be your u because you don't know its integral!

- Area under a curve is given by $\int_a^b y\,dx$

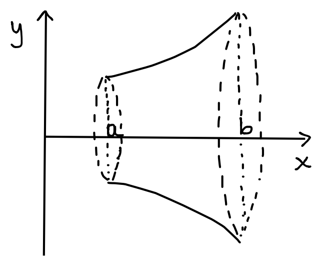


In the case of parametric equations

$$\text{Area} = \int_{t_1}^{t_2} y \frac{dx}{dt} \cdot dt$$

where t_1 and t_2 are the limits in terms of t

- Volume of revolution formed when curve is rotated through 2π radians about the x-axis is given by $\pi \int_a^b y^2\,dx$



In the case of parametric equations

$$\text{Volume} = \pi \int_{t_1}^{t_2} y^2 \frac{dx}{dt} \cdot dt$$

where t_1 and t_2 are the limits in terms of t

- Trapezium rule: This is a numerical method that yields an estimate for the integral.

$$\int_a^b y\,dx \approx \frac{1}{2} h \left\{ y_0 + 2(y_1 + \dots + y_{n-1}) + y_n \right\} \quad \text{where } h = \frac{b-a}{n}$$

Hint: You can generate the table for the trapezium rule using your calculator

$\boxed{\text{MODE}}$ $\boxed{\text{3:TABLE}}$ and proceed

$$\text{Percentage error} = \frac{|\text{Exact} - \text{estimate}|}{\text{Exact}} \times 100\%$$

- Differential equations: These are solved by separating the variables, ie collect all terms in y with dy on one side and all terms in x with dx on the other side and then integrate both sides.

When separating the variables (ie rearranging the differential equation) remember that you can only MULTIPLY OR DIVIDE, NOT ADD OR SUBTRACT.

If boundary conditions are given, then a particular solution can be found.