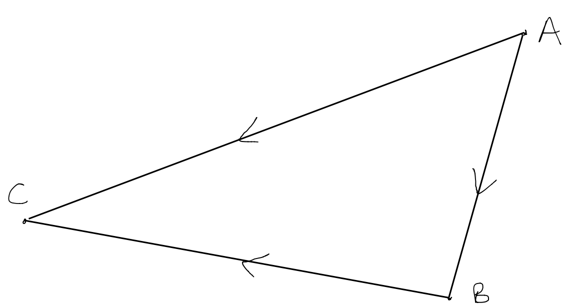
C4 - Chapter 5 - Vectors

* A scalar quantity has only magnitude
* A vector quantity has both magnitude and direction
* Vectors that are equal have both the same magnitude and same direction.
* Any vector parallel to the vector $\underset{\sim}{a}$ may be written as $\lambda\underset{\sim}{a}$, where λ is a non-zero scalar.

  e.g. Suppose $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}$ $\quad \underset{\sim}{b} = 3\underset{\sim}{i} + 3\underset{\sim}{j}$

  Then $\underset{\sim}{b} = 3\underset{\sim}{a} \Rightarrow \underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel vectors

* Vectors can be added using the "triangle law".



End

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

Start

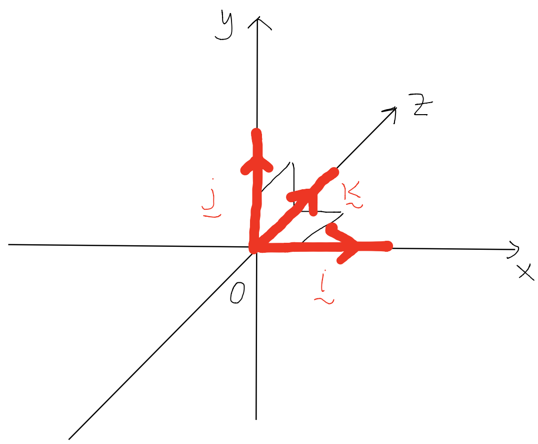
When you break up the journey into two or more legs, make sure that each subsequent leg starts from where you stopped

* The modulus (or magnitude, or length, or distance) of vector $\underset{\sim}{a}$ is written as $|\underset{\sim}{a}|$.
* A unit vector is a vector which has magnitude 1 unit

  Note: For any vector $\underset{\sim}{a}$, the vector $\dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$ is a unit vector in the direction of $\underset{\sim}{a}$.

* If $\lambda\underset{\sim}{a} + \mu\underset{\sim}{b} = \gamma\underset{\sim}{a} + \delta\underset{\sim}{b}$, and the non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are not parallel, then $\lambda = \gamma$ and $\mu = \delta$.
* The position vector of a point A is the vector $\overrightarrow{OA}$ where O is the origin. $\overrightarrow{OA}$ is usually written as $\underset{\sim}{a}$.
* $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA} = \underset{\sim}{b} - \underset{\sim}{a}$

* The vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors parallel to the x, y and z axes, and in the direction of x increasing, y increasing and z increasing respectively.



* Vectors with Cartesian Components can be written in column matrix form.

ie $\quad x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

* The modulus of $x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$ is given by $\sqrt{x^2+y^2+z^2}$. This corresponds to the magnitude of the position vector of the point with coordinates (x, y, z) and is the same as the distance between the point and the origin.

* The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$$

* The scalar product of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is defined by

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\ |\underset{\sim}{b}| \cos\theta$$

where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

It follows that: ① If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel then

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\ |\underset{\sim}{b}| \qquad (\text{since } \cos 0 = 1)$$

② If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular then

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = 0 \qquad (\text{since } \cos 90 = 0)$$

* If $\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

* A vector equation of the straight line passing through A and B, with position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively is given by

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a})$$ where λ is a scalar parameter

e.g. If A and B have coordinates (2,3,4) and (-1,-2,9) respectively then to find the equation of the straight line passing through A and B first start by finding $\vec{AB}$

$$\vec{AB} = \begin{pmatrix} -1 \\ -2 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ -5 \\ 5 \end{pmatrix}$$

$$\therefore \underset{\sim}{r} = \underbrace{2\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}}_{\text{Position vector}} + \lambda\big(\underbrace{-3\underset{\sim}{i} - 5\underset{\sim}{j} + 5\underset{\sim}{k}}_{\text{Direction vector}}\big)$$

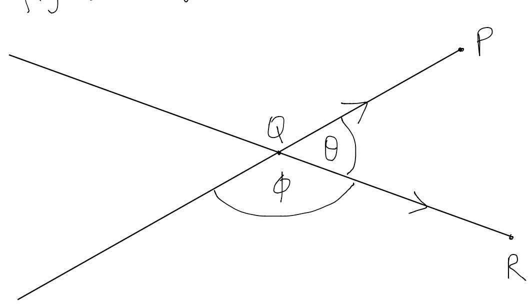
$$= \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -5 \\ 5 \end{pmatrix}$$

Both of these forms are acceptable

* The acute angle between two lines with direction vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, respectively is given by

$$\cos\theta = \left| \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|} \right|$$

* NOTE: There are always two angles between two lines, as shown in the figure below.

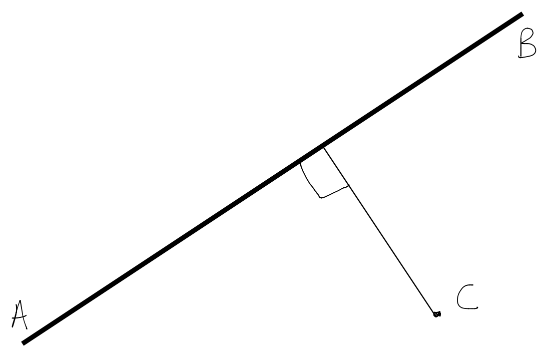


Obviously $\theta + \phi = 180^\circ$

In order to make sure that you are finding the correct angle check that both vectors point towards the same direction (either both inwards or both outwards

In this example, if I use $\overrightarrow{QP}$ and $\overrightarrow{QR}$ I will find θ. But if I use $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ I will find ϕ.

* NOTE: The shortest distance between a point and a line is always going to be the perpendicular distance, as shown below.



Note that in such cases, you will be working in a right-angled triangle.

* NOTE: If you are asked to show that two lines intersect then you start by equating the i, j and k components of the two lines that will lead to 3 equations in 2 unknowns.

Pick any of the two equations to solve simultaneously and then check that the values you have found also satisfy the third one.

Don't forget to write a conclusion (e.g. "Therefore the lines intersect").

$$\underbrace{\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \quad 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}}_{\text{VECTORS}} \qquad \underbrace{(2, 1, -3)}_{\text{COORDINATES}}$$

Don't mix them up!