June 05 Q8. Liquid is pouring into a container at a constant rate of 20 $\text{cm}^3\ \text{s}^{-1}$ and is leaking out at a rate proportional to the volume of the liquid already in the container.

(*a*) Explain why, at time *t* seconds, the volume, V cm^3, of liquid in the container satisfies the differential equation

$$\frac{dV}{dt} = 20 - kV,$$

where *k* is a positive constant.

(2)

The container is initially empty.

(*b*) By solving the differential equation, show that

$$V = A + Be^{-kt},$$

giving the values of *A* and *B* in terms of *k*.

(6)

Given also that $\frac{dV}{dt} = 10$ when $t = 5$,

(*c*) find the volume of liquid in the container at 10 s after the start.

(5)

Jan 06 Q7. The volume of a spherical balloon of radius *r* cm is V cm^3, where $V = \frac{4}{3}\pi r^3$.

(*a*) Find $\frac{dV}{dr}$.

(1)

The volume of the balloon increases with time *t* seconds according to the formula

$$\frac{dV}{dt} = \frac{1000}{(2t+1)^2}, \quad t \geq 0.$$

(*b*) Using the chain rule, or otherwise, find an expression in terms of *r* and *t* for $\frac{dr}{dt}$.

(2)

(*c*) Given that $V = 0$ when $t = 0$, solve the differential equation $\frac{dV}{dt} = \frac{1000}{(2t+1)^2}$, to obtain *V* in terms of *t*.

(4)

(*d*) Hence, at time $t = 5$,

(i) find the radius of the balloon, giving your answer to 3 significant figures,

(3)

(ii) show that the rate of increase of the radius of the balloon is approximately 2.90×10^{-2} cm s^{-1}.

(2)

June 06 Q7.

At time t seconds the length of the side of a cube is x cm, the surface area of the cube is S cm^2, and the volume of the cube is V cm^3.

The surface area of the cube is increasing at a constant rate of 8 cm^2 s^{-1}.

Show that

(*a*) $\dfrac{dx}{dt} = \dfrac{k}{x}$, where k is a constant to be found,

(4)

(*b*) $\dfrac{dV}{dt} = 2V^{\frac{1}{3}}$.

(4)

Given that $V = 8$ when $t = 0$,

(*c*) solve the differential equation in part (*b*), and find the value of t when $V = 16\sqrt{2}$.

(7)

Jan 07 Q4. (*a*) Express $\dfrac{2x-1}{(x-1)(2x-3)}$ in partial fractions.

(3)

(*b*) Given that $x \geq 2$, find the general solution of the differential equation

$$(2x-3)(x-1)\frac{dy}{dx} = (2x-1)y.$$

(5)

(*c*) Hence find the particular solution of this differential equation that satisfies $y = 10$ at $x = 2$, giving your answer in the form $y = f(x)$.

(4)

June 07 Q8. A population growth is modelled by the differential equation

$$\frac{dP}{dt} = kP,$$

where P is the population, t is the time measured in days and k is a positive constant.

Given that the initial population is P_0,

(*a*) solve the differential equation, giving P in terms of P_0, k and t.

(4)

Given also that $k = 2.5$,

(*b*) find the time taken, to the nearest minute, for the population to reach $2P_0$.

(3)

In an improved model the differential equation is given as

$$\frac{dP}{dt} = \lambda P \cos \lambda t,$$

where P is the population, t is the time measured in days and λ is a positive constant.

Given, again, that the initial population is P_0 and that time is measured in days,

(*c*) solve the second differential equation, giving P in terms of P_0, λ and t.

(4)

Given also that $\lambda = 2.5$,

(*d*) find the time taken, to the nearest minute, for the population to reach $2P_0$ for the first time, using the improved model.

(3)

Jan 08 Q8. Liquid is pouring into a large vertical circular cylinder at a constant rate of 1600 cm^3s^{-1} and is leaking out of a hole in the base, at a rate proportional to the square root of the height of the liquid already in the cylinder. The area of the circular cross section of the cylinder is 4000 cm^2.

(*a*) Show that at time t seconds, the height h cm of liquid in the cylinder satisfies the differential equation

$$\frac{dh}{dt} = 0.4 - k\sqrt{h},$$

where k is a positive constant.

(3)

When $h = 25$, water is leaking out of the hole at 400 cm^3s^{-1}.

(*b*) Show that $k = 0.02$.

(1)

(*c*) Separate the variables of the differential equation

$$\frac{dh}{dt} = 0.4 - 0.02\sqrt{h}$$

to show that the time taken to fill the cylinder from empty to a height of 100 cm is given by

$$\int_0^{100} \frac{50}{20 - \sqrt{h}}\, dh\,.$$

(2)

Using the substitution $h = (20 - x)^2$, or otherwise,

(*d*) find the exact value of $\int_0^{100} \frac{50}{20 - \sqrt{h}}\, dh$.

(6)

(*e*) Hence find the time taken to fill the cylinder from empty to a height of 100 cm, giving your answer in minutes and seconds to the nearest second.

(1)

June 08 Q3.

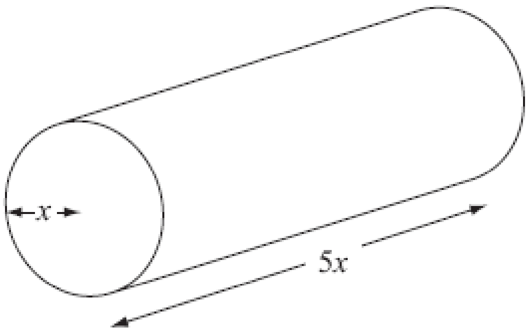

Figure 2

Figure 2 shows a right circular cylindrical metal rod which is expanding as it is heated. After t seconds the radius of the rod is x cm and the length of the rod is $5x$ cm.

The cross-sectional area of the rod is increasing at the constant rate of $0.032\ \text{cm}^2\,\text{s}^{-1}$.

(*a*) Find $\frac{dx}{dt}$ when the radius of the rod is 2 cm, giving your answer to 3 significant figures.

(4)

(*b*) Find the rate of increase of the volume of the rod when $x = 2$.

(4)

Jan 09 Q5.

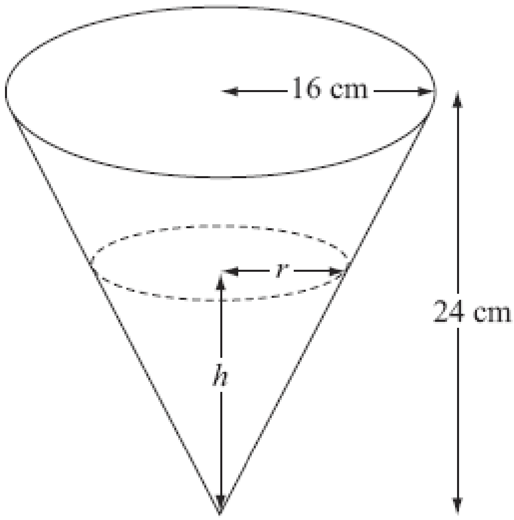

Figure 2

A container is made in the shape of a hollow inverted right circular cone. The height of the container is 24 cm and the radius is 16 cm, as shown in Figure 2. Water is flowing into the container. When the height of water is h cm, the surface of the water has radius r cm and the volume of water is V cm^3.

(*a*) Show that $V = \dfrac{4\pi h^3}{27}$.

(2)

[*The volume V of a right circular cone with vertical height h and base radius r is given by the formula* $V = \dfrac{1}{3}\pi r^2 h$.]

Water flows into the container at a rate of 8 cm^3 s^{-1}.

(*b*) Find, in terms of π, the rate of change of h when $h = 12$.

(5)

Jan 10 Q5. (*a*) Find $\displaystyle\int \frac{9x+6}{x}\,dx$, $x > 0$.

(2)

(*b*) Given that $y = 8$ at $x = 1$, solve the differential equation

$$\frac{dy}{dx} = \frac{(9x+6)y^{\frac{1}{3}}}{x}$$

giving your answer in the form $y^2 = g(x)$.

(6)

Jan 10 Q6. The area A of a circle is increasing at a constant rate of 1.5 cm^2 s^{-1}. Find, to 3 significant figures, the rate at which the radius r of the circle is increasing when the area of the circle is 2 cm^2.

(5)

June 10 Q8.

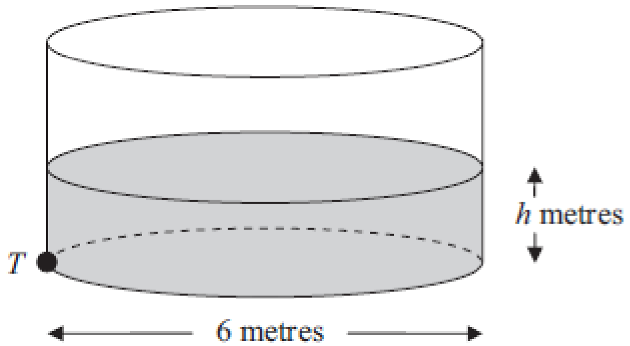

Figure 2

Figure 2 shows a cylindrical water tank. The diameter of a circular cross-section of the tank is 6 m. Water is flowing into the tank at a constant rate of 0.48π m^3 min^{-1}. At time t minutes, the depth of the water in the tank is h metres. There is a tap at a point T at the bottom of the tank. When the tap is open, water leaves the tank at a rate of $0.6\pi h$ m^3 min^{-1}.

(*a*) Show that, t minutes after the tap has been opened,

$$75\frac{dh}{dt} = (4 - 5h).$$

(5)

When $t = 0$, $h = 0.2$

(*b*) Find the value of t when $h = 0.5$

(6)

Jan 11 Q3. (*a*) Express $\frac{5}{(x-1)(3x+2)}$ in partial fractions.

(3)

(*b*) Hence find $\int \frac{5}{(x-1)(3x+2)}\,dx$, where $x > 1$.

(3)

(*c*) Find the particular solution of the differential equation

$$(x - 1)(3x + 2)\frac{dy}{dx} = 5y, \quad x > 1,$$

for which $y = 8$ at $x = 2$. Give your answer in the form $y = f(x)$.

(6)

June 11 Q3.

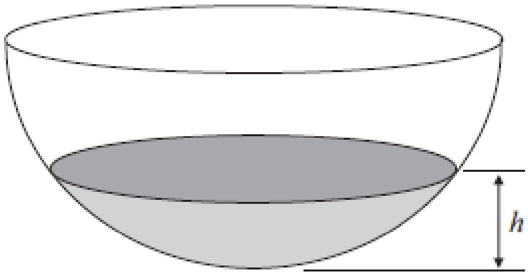

Figure 1

A hollow hemispherical bowl is shown in Figure 1. Water is flowing into the bowl.

When the depth of the water is h m, the volume V m^3 is given by

$$V = \frac{1}{12}\pi h^2(3 - 4h), \qquad 0 \le h \le 0.25.$$

(a) Find, in terms of π, $\dfrac{dV}{dh}$ when $h = 0.1$.

(4)

Water flows into the bowl at a rate of $\dfrac{\pi}{800}$ m^3 s^{-1}.

(b) Find the rate of change of h, in m s^{-1}, when $h = 0.1$.

(2)

June 11 Q8. (a) Find $\displaystyle\int (4y+3)^{-\frac{1}{2}}\, dy$.

(2)

(b) Given that y =1.5 at x = – 2, solve the differential equation

$$\frac{dy}{dx} = \frac{\sqrt{(4y+3)}}{x^2},$$

giving your answer in the form y = f(x).

(6)

Jan 12 Q8. (*a*) Express $\dfrac{1}{P(5-P)}$ in partial fractions.

(3)

A team of conservationists is studying the population of meerkats on a nature reserve. The population is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{15}P(5-P), \quad t \geq 0,$$

where P, in thousands, is the population of meerkats and t is the time measured in years since the study began.

Given that when $t = 0$, $P = 1$,

(*b*) solve the differential equation, giving your answer in the form,

$$P = \frac{a}{b + c\mathrm{e}^{-\frac{1}{3}t}}$$

where a, b and c are integers.

(8)

(*c*) Hence show that the population cannot exceed 5000.

(1)