C4 - JANUARY 2011

1. $\int_0^{\pi/2} x\sin 2x \, dx = \left[-\frac{1}{2}x\cos 2x\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{1}{2}\cos 2x \, dx$

$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

$= -\frac{1}{2}\frac{\pi}{2}(-1) + \left[\frac{1}{4}\sin 2x\right]_0^{\pi/2}$

$u = x \qquad \frac{dv}{dx} = \sin 2x$

$= \frac{\pi}{4}$

$\frac{du}{dx} = 1 \qquad v = -\frac{1}{2}\cos 2x$

2. $I = 16 - 16(0.5)^t$

If $y = a^x$ then $\frac{dy}{dx} = a^x \ln a$ IMPLICIT DIFFERENTIATION

$\frac{dI}{dt} = -16\left(0.5^t\right)\ln 0.5$

$\left.\frac{dI}{dt}\right|_{t=3} = -16(0.5)^3 \ln 0.5 = -2\ln 0.5 = \ln 4$

* $y = a^x \Rightarrow \ln y = \ln a^x \Rightarrow \ln y = x\ln a$

$\frac{1}{y}\frac{dy}{dx} = \ln a \Rightarrow \frac{dy}{dx} = y\ln a = a^x \ln a$

3 a) $\dfrac{5}{(x-1)(3x+2)} = \dfrac{A}{x-1} + \dfrac{B}{3x+2} \Rightarrow 5 = A(3x+2) + B(x-1)$

$x=1 \quad 5 = 5A \Rightarrow A = 1$

$x = -2/3 \quad 5 = -5/3\, B \Rightarrow B = -3$

$$\frac{5}{(x-1)(3x+2)} = \frac{1}{x-1} - \frac{3}{3x+2}$$

b) $\displaystyle\int \frac{5}{(x-1)(3x+2)}\,dx = \int \frac{1}{x-1} - \frac{3}{3x+2}\,dx = \ln(x-1) - \frac{3}{3}\ln(3x+2) + C$

c) $(x-1)(3x+2)\dfrac{dy}{dx} = 5y$

$\displaystyle\int \frac{1}{y}\,dy = \int \frac{5}{(x-1)(3x+2)}\,dx$

Good practice to write C as lnA as it makes manipulation of ln's easier

$\ln y = \ln(x-1) - \ln(3x+2) + \ln A$

$x=2, y=8 \Rightarrow \ln 8 = \ln 1 - \ln 8 + \ln A$

$2\ln 8 = \ln A$

$A = 64$

$\ln y = \ln(x-1) - \ln(3x+2) + \ln 64 \Rightarrow \ln y = \ln\left(\dfrac{64(x-1)}{3x+2}\right) \Rightarrow y = \dfrac{64(x-1)}{3x+2}$

4 a) $\overrightarrow{AB} = \begin{pmatrix}-2\\2\\-1\end{pmatrix} - \begin{pmatrix}1\\-3\\2\end{pmatrix} = \begin{pmatrix}-3\\5\\-3\end{pmatrix}$

b) $l: \underline{r} = \begin{pmatrix}1\\-3\\2\end{pmatrix} + \lambda \begin{pmatrix}-3\\5\\-3\end{pmatrix}$

c) $\overrightarrow{AC} = \begin{pmatrix}2\\p\\-4\end{pmatrix} - \begin{pmatrix}1\\-3\\2\end{pmatrix} = \begin{pmatrix}1\\p+3\\-6\end{pmatrix}$ $\quad \begin{pmatrix}1\\p+3\\-6\end{pmatrix} \cdot \begin{pmatrix}-3\\5\\-3\end{pmatrix} = 0$

$-3 + 5(p+3) + 18 = 0$

$p = -6$

d) $A(1,-3,2)$ $\quad C(2,-6,-4)$

$AC = \sqrt{(1-2)^2 + (-3-(-6))^2 + (2-(-4))^2} = \sqrt{46}$

5a) $(2-3x)^{-2} = 2^{-2}\left(1 - \frac{3x}{2}\right)^{-2}$

$= \frac{1}{4}\left\{1 + (-2)\left(-\frac{3x}{2}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{3x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(-\frac{3x}{2}\right)^3 + \dots\right\}$

$= \frac{1}{4} + \frac{3x}{4} + \frac{27x^2}{16} + \frac{27x^3}{8} + \dots$

b) $(a+bx)\left(\frac{1}{4} + \frac{3}{4}x + \frac{27x^2}{16} + \frac{27x^3}{8}\right)$

Coefficient of x: $\frac{3}{4}a + \frac{1}{4}b = 0 \quad \Rightarrow 3a + b = 0 \quad [1]$

Coefficient of x^2: $\frac{27}{16}a + \frac{3}{4}b = \frac{9}{16} \quad \Rightarrow 27a + 12b = 9$

$9a + 4b = 3 \quad [2]$

$$9a + 4(-3a) = 3$$
$$-3a = 3$$
$$a = -1$$
$$b = 3$$

c) Coefficient of x^3: $\frac{27}{8}a + \frac{27}{16}b = -\frac{27}{8} + \frac{81}{16} = \frac{27}{16}$

6 a) $\frac{dx}{dt} = \frac{1}{t}$ $\quad \frac{dy}{dt} = 2t$ $\quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1/t} = 2t^2$

$\left.\frac{dy}{dx}\right|_{t=3} = 2 \cdot 3^2 = 18$ $\quad m_{NORMAL} = -\frac{1}{18}$

When $t=3$, $x = \ln 3$, $y = 7$ $\quad y - 7 = -\frac{1}{18}(x - \ln 3)$

b) $x = \ln t \Rightarrow t = e^x$ $\quad y = (e^x)^2 - 2$

$$y = e^{2x} - 2$$

c) $V = \pi \int_{\ln 2}^{\ln 4} y^2 \, dx = \pi \int_{\ln 2}^{\ln 4} (e^{2x} - 2)^2 \, dx$

$= \pi \int_{\ln 2}^{\ln 4} (e^{4x} - 4e^{2x} + 4) \, dx$

$= \pi \left[\frac{1}{4} e^{4x} - \frac{4}{2} e^{2x} + 4x \right]_{\ln 2}^{\ln 4}$

$= \pi \left\{ (64 - 32 + 4\ln 4) - (4 - 8 + 4\ln 2) \right\}$

$= \pi (36 + 4\ln 2)$

7 a) When $x = 3$, $y = 0.1847$ When $x = 5$, $y = 0.1667$

b) $I \approx \frac{1}{2}(1) \left\{ 0.2 + 2(0.1847 + 0.1745) + 0.1667 \right\}$

$= 0.543$

c) $x = (u-4)^2 + 1$

$dx = 2(u-4) \, du$

$\sqrt{x-1} + 4 = u$

When $x = 2$, $u = 5$

When $x = 5$, $u = 6$

$I = \int_5^6 \frac{1}{u} \, 2(u-4) \, du$

$= \int_5^6 2 - \frac{8}{u} \, du = \left[2u - 8\ln u \right]_5^6$

$= 2 - 8\ln(6/5)$