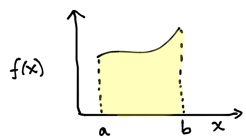
S2 - Chapter 3 - Continuous random variables - Summary

* For a given function $f(x)$ to be a valid probability density function
  - $f(x) \geqslant 0$ (we cannot have negative probabilities)
  - $\int f(x)\,dx = 1$ (since "sum" of probabilities is 1)

* 
$P(a<X<b) = \int_a^b f(x)\,dx$ (the shaded area)

* $f(x) = \frac{d}{dx}\left[F(x)\right]$
* $F(x) = \int^x f(t)\,dt$

* $\mu = E(X) = \int x f(x)\,dx$
* $\sigma^2 = Var(X) = E(X^2) - \left[E(X)\right]^2 = \int x^2 f(x)\,dx - \mu^2$

* The mode is that value of X that leads to the peak (maximum) of $f(x)$. If it is not easy to spot the maximum then differentiate $f(x)$ to find it. If $f(x)$ is simply a straight line then the mode will be found either at the start or the end.
* Median, m, satisfies $F(m) = 0.5$
* Lower quartile, Q_1, satisfies $F(Q_1) = 0.25$
* Upper quartile, Q_3, satisfies $F(Q_3) = 0.75$

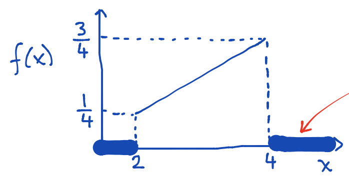
* Example: $f(x) = \begin{cases} K(x-1) & 2 < x < 4 \\ 0 & \text{otherwise} \end{cases}$

i) Find K
ii) Sketch $f(x)$
iii) Find $F(x)$
iv) Find $E(X)$
v) Find the mode and the median.

i) $\int_2^4 K(x-1)\,dx = K\left[\frac{x^2}{2} - x\right]_2^4 = K\left[\left(\frac{4^2}{2} - 4\right) - \left(\frac{2^2}{2} - 2\right)\right]$

$= 4K = 1 \implies K = 1/4$

ii)



Do not forget bold lines to the left and right.

iii) $\int_2^x \frac{1}{4}(t-1)\,dt = \frac{1}{4}\left[\frac{t^2}{2} - t\right]_2^x = \frac{1}{4}\left[\left(\frac{x^2}{2} - x\right) - \left(\frac{2^2}{2} - 2\right)\right] = \frac{1}{4}\left(\frac{x^2}{2} - x\right)$

$$F(x) = \begin{cases} 0 & x < 2 \\ \frac{1}{4}\left(\frac{x^2}{2} - x\right) & 2 \leq x \leq 4 \\ 1 & x > 4 \end{cases}$$

iv) $E(X) = \int_2^4 x \cdot \frac{1}{4}(x-1) = \int_2^4 x^2/4 - x/4\,dx = \left[\frac{x^3}{12} - \frac{x^2}{8}\right]_2^4 = \left(\frac{4^3}{12} - \frac{4^2}{8}\right) - \left(\frac{2^3}{12} - \frac{2^2}{8}\right)$

$= \frac{19}{6}$

v) Mode = 4 (by observation).
Median, m, satisfies $F(m) = 0.5$

$\implies \frac{1}{4}\left(\frac{m^2}{2} - m\right) = \frac{1}{2}$

$\frac{m^2}{8} - \frac{m}{4} - \frac{1}{2} = 0$

$m^2 - 2m - 4 = 0$

$m = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-4)}}{2}$

$= 1 \pm \sqrt{5}$

Reject $1 - \sqrt{5}$ as it lies outside the range of x
Hence $m = 1 + \sqrt{5}$